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The sixth Painlevé equation arising from $D_4^{(1)}$ hierarchy

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Abstract
The sixth Painlevé equation arises from a Drinfeld–Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

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1. Introduction

The Drinfeld–Sokolov hierarchies are extensions of the KdV (or mKdV) hierarchy [DS]. It is known that their similarity reductions imply several Painlevé equations [AS, KK1, NY1]. For the sixth Painlevé equation ($P_{\rm VI}$), the relation with the $A_2^{(1)}$-type hierarchy is investigated [KK2]. On the other hand, $P_{\rm VI}$ admits a group of symmetries which is isomorphic to the affine Weyl group of type $D_4^{(1)}$ [O]. Also it is known that $P_{\rm VI}$ is derived from the Lax pair associated with the algebra $\widehat{\mathfrak{so}}(8)$ [NY3]. However, the relation between $D_4^{(1)}$-type hierarchies and $P_{\rm VI}$ has not been clarified. In this paper, we show that the sixth Painlevé equation is derived from a Drinfeld–Sokolov hierarchy of type $D_4^{(1)}$ by similarity reduction.

Consider a Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_2(x)y = 0, \tag{1.1}$$

with the Riemann scheme

$$\left\{\begin{matrix} x=t_0 & x=t_1 & x=t_3 & x=t_4 & x=\lambda & x=\infty \\ 0 & 0 & 0 & 0 & 0 & \rho \\ \theta_0 & \theta_1 & \theta_3 & \theta_4 & 2 & \rho+1 \end{matrix}\right\},$$

satisfying the relation

$$\theta_0 + \theta_1 + \theta_3 + \theta_4 + 2\rho = 1.$$

We also let $\mu = \mathrm{Res}_{x=\lambda}\, p_2(x)\,\mathrm{d}x$. Then the monodromy preserving deformation of the equation (1.1) is described as a system of partial differential equations for λ and μ. This system can be regarded as the symmetric representation of P_{VI} [Kaw]. We discuss a derivation of the symmetric representation in the case

$$t_0 = -t \qquad t_1 = -\frac{t+1}{t-1} \qquad t_3 = \frac{t-1}{t+1} \qquad t_4 = \frac{1}{t}$$
$$\theta_0 = \alpha_0 \qquad \theta_1 = \alpha_1 - 1 \qquad \theta_3 = \alpha_3 - 1 \qquad \theta_4 = \alpha_4 - 1 \qquad \rho = \alpha_2.$$

Note that

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 4.$$

With the notation

$$F_0 = \lambda + t, \qquad F_1 = \lambda + \frac{t+1}{t-1}, \qquad F_2 = \mu, \qquad F_3 = \lambda - \frac{t-1}{t+1}, \qquad F_4 = \lambda - \frac{1}{t},$$

the dependence of λ and μ on t is given by

$$\begin{aligned}\vartheta(F_j) = {} & 2F_0F_1F_2F_3F_4 - (\alpha_0 - 1)F_1F_3F_4 \\ & - (\alpha_1 - 1)F_0F_3F_4 - (\alpha_3 - 1)F_0F_1F_4 - (\alpha_4 - 1)F_0F_1F_3 + \Theta_j, \end{aligned} \tag{1.2}$$

for $j = 0, 1, 3, 4$ and

$$\begin{aligned}\vartheta(F_2) = {} & -F_2^2(F_0F_1F_3 + F_0F_1F_4 + F_0F_3F_4 + F_1F_3F_4) \\ & + F_2\{(\alpha_3 + \alpha_4 - 2)F_0F_1 + (\alpha_1 + \alpha_4 - 2)F_0F_3 + (\alpha_1 + \alpha_3 - 2)F_0F_4 \\ & + (\alpha_0 + \alpha_4 - 2)F_1F_3 + (\alpha_0 + \alpha_3 - 2)F_1F_4 + (\alpha_0 + \alpha_1 - 2)F_3F_4\} \\ & - \alpha_2\{(\alpha_0 + \alpha_2 - 1)F_0 + (\alpha_1 + \alpha_2 - 1)F_1 + (\alpha_3 + \alpha_2 - 1)F_3 \\ & + (\alpha_4 + \alpha_2 - 1)F_4\}, \end{aligned} \tag{1.3}$$

where

$$\vartheta = \Theta_0 \frac{\mathrm{d}}{\mathrm{d}t}, \qquad \Theta_i = \prod_{j=0,1,3,4;\, j\neq i} (F_i - F_j).$$

Note that the system (1.2), (1.3) is equivalent to the Hamiltonian system:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\partial H'}{\partial \mu}, \qquad \frac{\mathrm{d}\mu}{\mathrm{d}t} = -\frac{\partial H'}{\partial \lambda}, \tag{1.4}$$

where the Hamiltonian $H' = H'(\lambda, \mu, t)$ is given by

$$\begin{aligned}\Theta_0 H' = {} & F_0F_1F_2^2F_3F_4 - (\alpha_0 - 1)F_1F_2F_3F_4 - (\alpha_1 - 1)F_0F_2F_3F_4 \\ & - (\alpha_3 - 1)F_0F_1F_2F_4 - (\alpha_4 - 1)F_0F_1F_2F_3 + \alpha_2F_0\{(\alpha_0 - 1)F_0 \\ & + (\alpha_1 + \alpha_2 - 1)F_1 + (\alpha_3 + \alpha_2 - 1)F_3 + (\alpha_4 + \alpha_2 - 1)F_4\}. \end{aligned}$$

We also remark that the system (1.4) is transformed into the Hamiltonian system for P_{VI} as in [IKSY]

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial q},$$

with the Hamiltonian

$$\begin{aligned}s(s-1)H = {} & q(q-1)(q-s)p^2 - \tfrac{1}{4}\{(\alpha_1 - 4)q(q-1) \\ & + \alpha_3 q(q-s) + \alpha_4(q-1)(q-s)\}p + \tfrac{1}{16}\alpha_2(\alpha_0 + \alpha_2)q, \end{aligned}$$

by the canonical transformation $(\lambda, \mu, t, H') \to (q, p, s, H)$ defined as

$$q = \frac{\left(t + \frac{t-1}{t+1}\right)F_4}{\left(\frac{t-1}{t+1} - \frac{1}{t}\right)F_0}, \qquad p = \frac{\left(\frac{t-1}{t+1} - \frac{1}{t}\right)F_0(F_0F_2 + \alpha_2)}{4\left(t + \frac{t-1}{t+1}\right)\left(t + \frac{1}{t}\right)},$$

and

$$s = -\frac{\left(t + \frac{t-1}{t+1}\right)\left(\frac{t+1}{t-1} + \frac{1}{t}\right)}{\left(t - \frac{t+1}{t-1}\right)\left(\frac{t-1}{t+1} - \frac{1}{t}\right)}.$$

This paper is organized as follows. In section 2, we recall the definition of the affine Lie algebra $\mathfrak{g} = \mathfrak{g}(D_4^{(1)})$. In section 3, a Drinfeld–Sokolov hierarchy of type $D_4^{(1)}$ is formulated. In sections 4 and 5, we show that its similarity reduction implies the symmetric representation of P_{VI}.

2. Affine Lie algebra

In the notation of [Kac], the affine Lie algebra $\mathfrak{g} = \mathfrak{g}\big(D_4^{(1)}\big)$ is the Lie algebra generated by the Chevalley generators e_i, f_i, $\alpha_i^\vee$ $(i = 0, \ldots, 4)$ and the scaling element d with the fundamental relations

$$\begin{aligned}
&(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0, \qquad (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0 \qquad (i \neq j),\\
&\left[\alpha_i^\vee, \alpha_j^\vee\right] = 0, \qquad \left[\alpha_i^\vee, e_j\right] = a_{ij} e_j, \qquad \left[\alpha_i^\vee, f_j\right] = -a_{ij} f_j, \qquad [e_i, f_j] = \delta_{i,j}\alpha_i^\vee,\\
&\left[d, \alpha_i^\vee\right] = 0, \qquad [d, e_i] = \delta_{i,0} e_0, \qquad [d, f_i] = -\delta_{i,0} f_0,
\end{aligned}$$

for $i, j = 0, \ldots, 4$, where $A = (a_{ij})_{i,j=0}^4$ is the generalized Cartan matrix of type $D_4^{(1)}$ defined by

$$A = \begin{Bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{Bmatrix}.$$

We denote the Cartan subalgebra of $\mathfrak{g}$ by

$$\mathfrak{h} = \bigoplus_{j=0}^{4} \mathbb{C}\alpha_j^\vee \oplus \mathbb{C}d.$$

The canonical central element of $\mathfrak{g}$ is given by

$$K = \alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee.$$

The normalized invariant form $(|) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is determined by the conditions

$$\begin{aligned}
&\left(\alpha_i^\vee|\alpha_j^\vee\right) = a_{ij}, \qquad (e_i|f_j) = \delta_{i,j}, \qquad \left(\alpha_i^\vee|e_j\right) = \left(\alpha_i^\vee|f_j\right) = 0,\\
&(d|d) = 0, \qquad \left(d|\alpha_j^\vee\right) = \delta_{0,j}, \qquad (d|e_j) = (d|f_j) = 0,
\end{aligned}$$

for $i, j = 0, \ldots, 4$.

We consider the $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(s)$ of type $s = (1, 1, 0, 1, 1)$ by setting

$$\deg \mathfrak{h} = \deg e_2 = \deg f_2 = 0, \qquad \deg e_i = 1, \qquad \deg f_i = -1 \qquad (i = 0, 1, 3, 4).$$

If we take an element $d_s \in \mathfrak{h}$ such that

$$\left(d_s|\alpha_2^\vee\right) = 0, \qquad \left(d_s|\alpha_j^\vee\right) = 1 \qquad (j = 0, 1, 3, 4),$$

this gradation is defined by

$$\mathfrak{g}_k(s) = \{x \in \mathfrak{g} \mid [d_s, x] = kx\} \qquad (k \in \mathbb{Z}).$$

In the following, we choose

$$d_s = 4d + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_4^\vee.$$

We set

$$\mathfrak{g}_{<0} = \bigoplus_{k<0} \mathfrak{g}_k(s), \qquad \mathfrak{g}_{\geqslant 0} = \bigoplus_{k\geqslant 0} \mathfrak{g}_k(s).$$

We choose the graded Heisenberg subalgebra $\mathfrak{s} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{s}_k(s)$ of $\mathfrak{g}$ of type $s = (1, 1, 0, 1, 1)$ with

$$\mathfrak{s}_1(s) = \mathbb{C}\Lambda_{1,1} \oplus \mathbb{C}\Lambda_{1,2},$$

where

$$\Lambda_{1,1} = -e_0 + e_1 + e_3 - e_{21} + e_{23} + e_{24}, \qquad \Lambda_{1,2} = e_1 - e_3 + e_4 + e_{20} + e_{21} + e_{23}.$$

Here we denote

$$e_{2j} = [e_2, e_j], \qquad f_{2j} = [f_2, f_j] \qquad (j = 0, 1, 3, 4).$$

We remark that

$$\mathfrak{s} = \{x \in \mathfrak{g} \mid [\Lambda_{1,1}, x] \in \mathbb{C}K\}.$$

and

$$\mathfrak{s}_0(s) = \mathbb{C}K, \qquad \mathfrak{s}_{2k}(s) = 0 \qquad (k \neq 0).$$

Each $\mathfrak{s}_{2k-1}(s)$ is expressed in the form

$$\mathfrak{s}_{2k-1}(s) = \mathbb{C}\Lambda_{2k-1,1} \oplus \mathbb{C}\Lambda_{2k-1,2},$$

with certain elements $\Lambda_{2k-1,i}$ $(i = 1, 2)$ satisfying

$$[\Lambda_{2k-1,i}, \Lambda_{2l-1,j}] = (2k-1)\delta_{i,j}\delta_{k+l,1}K \qquad (i, j = 1, 2; k, l \in \mathbb{Z}).$$

For $k = 0$, we have

$$\Lambda_{-1,1} = \tfrac{1}{2}(-2f_0 + f_1 + f_3 + f_{21} - f_{23} - 2f_{24}),$$

$$\Lambda_{-1,2} = \tfrac{1}{2}(f_1 - f_3 + 2f_4 - 2f_{20} - f_{21} - f_{23}).$$

Remark 2.1. In the notation of [C], the Heisenberg subalgebra $\mathfrak{s}$ corresponds to the conjugacy class $D_4(a_1)$ of the Weyl group $W(D_4)$; see [DF].

3. Drinfeld–Sokolov hierarchy

In the following, we use the notation of infinite-dimensional groups

$$G_{<0} = \exp(\widehat{\mathfrak{g}}_{<0}), \qquad G_{\geqslant 0} = \exp(\widehat{\mathfrak{g}}_{\geqslant 0}),$$

where $\widehat{\mathfrak{g}}_{<0}$ and $\widehat{\mathfrak{g}}_{\geqslant 0}$ are completions of $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{\geqslant 0}$, respectively.

Introducing the time variables $t_{k,i}$ $(i = 1, 2;\ k = 1, 3, 5, \ldots)$, we consider the *Sato equation* for a $G_{<0}$-valued function $W = W(t_{1,1}, t_{1,2}, \ldots)$

$$\partial_{k,i}(W) = B_{k,i}W - W\Lambda_{k,i} \qquad (i = 1, 2;\ k = 1, 3, 5, \ldots), \tag{3.1}$$

where $\partial_{k,i} = \partial/\partial t_{k,i}$ and $B_{k,i}$ stands for the $\mathfrak{g}_{\geqslant 0}$-component of $W\Lambda_{k,i}W^{-1} \in \widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geqslant 0}$. We understand the Sato equation (3.1) as a conventional form of the differential equation

$$\partial_{k,i} - B_{k,i} = W(\partial_{k,i} - \Lambda_{k,i})W^{-1} \qquad (i = 1, 2;\ k = 1, 3, 5, \ldots), \tag{3.2}$$

defined through the adjoint action of $G_{<0}$ on $\widehat{\mathfrak{g}}_{<0} \oplus \mathfrak{g}_{\geqslant 0}$. The Zakharov–Shabat equation,

$$[\partial_{k,i} - B_{k,i}, \partial_{l,j} - B_{l,j}] = 0 \qquad (i, j = 1, 2;\ k, l = 1, 3, 5, \ldots), \tag{3.3}$$

follows from the Sato equation (3.2).

The $\mathfrak{g}_{\geqslant 0}$-valued functions $B_{1,i}$ $(i = 1, 2)$ are expressed in the form

$$B_{1,i} = \Lambda_{1,i} + U_i, \qquad U_i = \sum_{j=0}^{4} u_{j,i}\alpha_j^\vee + x_i e_2 + y_i f_2. \tag{3.4}$$

The Zakharov–Shabat equation (3.3) for $k = 1$ is equivalent to

$$\partial_{1,i}(U_j) - \partial_{1,j}(U_i) + [U_j, U_i] = 0, \qquad [\Lambda_{1,i}, U_j] - [\Lambda_{1,j}, U_i] = 0, \tag{3.5}$$

for $i, j = 1, 2$. Then we have

Lemma 3.1. *Under the Sato equation (3.2), the following equations are satisfied:*

$$(d_s|\partial_{1,i}(U_j)) + \tfrac{1}{2}(U_i|U_j) = 0 \qquad (i, j = 1, 2). \tag{3.6}$$

Proof. The system (3.2) for $k = 1$ is equivalent to

$$\partial_{1,i} - \Lambda_{1,i} - U_i = W(\partial_{1,i} - \Lambda_{1,i})W^{-1} \qquad (i = 1, 2). \tag{3.7}$$

Set

$$W = \exp(w), \qquad w = \sum_{k=1}^{\infty} w_{-k}, \qquad w_{-k} \in \mathfrak{g}_{-k}(s).$$

Then the system (3.7) implies

$$U_i = \sum_{k=1}^{\infty} \frac{1}{k!}\,\mathrm{ad}(w)^{k-1}\partial_{1,i}(w) + \sum_{k=1}^{\infty} \frac{1}{k!}\,\mathrm{ad}(w)^{k}(\Lambda_{1,i}) \qquad (i = 1, 2). \tag{3.8}$$

Comparing the component of degree $-k$ in (3.8), we obtain

$$U_i = \mathrm{ad}(w_{-1})(\Lambda_{1,i}) \qquad (i = 1, 2),$$

for $k = 0$;

$$\mathrm{ad}(w_{-2})(\Lambda_{1,i}) + \tfrac{1}{2}\mathrm{ad}(w_{-1})^2(\Lambda_{1,i}) + \partial_{1,i}(w_{-1}) = 0 \qquad (i = 1, 2), \tag{3.9}$$

for $k = 1$;

$$\sum_{i_1+\cdots+i_l=k+1} \frac{1}{l!}\,\mathrm{ad}(w_{-i_1})\cdots\mathrm{ad}(w_{-i_l})(\Lambda_{1,i})$$
$$+ \sum_{i_1+\cdots+i_l=k} \frac{1}{l!}\,\mathrm{ad}(w_{-i_1})\cdots\mathrm{ad}(w_{-i_{l-1}})\partial_{1,i}(w_{-i_l}) = 0 \qquad (i = 1, 2),$$

for $k \geqslant 2$. On the other hand, we have

$$(\Lambda_{1,i}|\,\mathrm{ad}(\Lambda_{1,j})(x)) = 0 \qquad (i, j = 1, 2;\ x \in \mathfrak{g}_{-2}(s)),$$

and

$$(\Lambda_{1,i}|x) = (d_s|\,\mathrm{ad}(\Lambda_{1,i})(x)) \qquad (i = 1, 2;\ x \in \mathfrak{g}_{-1}(s)).$$

Hence it follows that

$$(\Lambda_{1,j}|\,\mathrm{LHS\ of\ (3.9)}) = \tfrac{1}{2}(\Lambda_{1,j}|\,\mathrm{ad}(w_{-1})^2(\Lambda_{1,i})) + (\Lambda_{1,j}|\partial_{1,i}(w_{-1}))$$
$$= -\tfrac{1}{2}(U_i|U_j) - (d_s|\partial_{1,i}(U_j)).$$

□

Remark 3.2. Let $X(0) \in G_{<0}G_{\geqslant 0}$ and define

$$X = X(t_{1,1}, t_{1,2}, \ldots) = \exp(\xi)X(0), \qquad \xi = \sum_{i=1,2}\sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$

Then a solution $W \in G_{<0}$ of the system (3.1) is given formally via the decomposition

$$X = W^{-1}Z, \qquad Z \in G_{\geqslant 0}.$$

4. Similarity reduction

Under the Sato equation (3.2), we consider the operator

$$\mathcal{M} = W \exp(\xi) d_s \exp(-\xi) W^{-1}, \qquad \xi = \sum_{i=1,2} \sum_{k=1,3,\ldots} t_{k,i} \Lambda_{k,i}.$$

Then the operator $\mathcal{M}$ satisfies

$$\partial_{k,i}(\mathcal{M}) = [B_{k,i}, \mathcal{M}] \qquad (i = 1, 2;\ k = 1, 3, 5, \ldots).$$

Note that

$$\mathcal{M} = d_s - \sum_{i=1,2} \sum_{k=1,3,\ldots} k t_{k,i} W \Lambda_{k,i} W^{-1} - d_s(W) W^{-1}.$$

Assuming that $t_{k,1} = t_{k,2} = 0$ for $k \geqslant 3$, we require that the similarity condition $\mathcal{M} \in \mathfrak{g}_{\geqslant 0}$ is satisfied. Then we have

$$\partial_{1,i}(\mathcal{M}) = [B_{1,i}, \mathcal{M}] \qquad (i = 1, 2).$$

where $\mathcal{M} = d_s - t_{1,1} B_{1,1} - t_{1,2} B_{1,2}$, or equivalently

$$[d_s - M, \partial_{1,i} - B_{1,i}] = 0 \qquad (i = 1, 2), \tag{4.1}$$

where $M = t_{1,1} B_{1,1} + t_{1,2} B_{1,2}$. Under the Zakharov–Shabat equation

$$[\partial_{1,1} - B_{1,1}, \partial_{1,2} - B_{1,2}] = 0,$$

the system (4.1) is equivalent to

$$\sum_{j=1,2} t_{1,j} \partial_{1,j}(B_{1,i}) = [d_s, B_{1,i}] - B_{1,i} \qquad (i = 1, 2).$$

In terms of the operators U_i, this similarity condition can be expressed as

$$\sum_{j=1,2} t_{1,j} \partial_{1,j}(U_i) + U_i = 0 \qquad (i = 1, 2). \tag{4.2}$$

We regard the systems (3.5), (3.6) and (4.2) as a similarity reduction of the Drinfeld–Sokolov hierarchy of type $D_4^{(1)}$.

In the notation (3.4), these systems are expressed in terms of the variables $u_{j,i}, x_i, y_i$ as follows:

$$\begin{aligned}
&\partial_{1,1}(x_2) - \partial_{1,2}(x_1) - (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2}) x_1 + (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2}) x_2 = 0, \\
&\partial_{1,1}(y_2) - \partial_{1,2}(y_1) + (u_{1,1} - u_{3,1} - u_{0,2} + u_{4,2}) y_1 - (u_{0,1} - u_{4,1} + u_{1,2} - u_{3,2}) y_2 = 0, \\
&\partial_{1,1}(u_{2,2}) - \partial_{1,2}(u_{2,1}) - x_1 y_2 + x_2 y_1 = 0, \\
&\partial_{1,1}(u_{j,2}) - \partial_{1,2}(u_{j,1}) = 0 \qquad (j = 0, 1, 3, 4),
\end{aligned}$$

and

$$\begin{aligned}
&u_{1,1} - 2u_{2,1} + u_{3,1} + 2u_{4,1} - u_{1,2} + u_{3,2} = 0, \\
&u_{1,1} - u_{3,1} - 2u_{0,2} - u_{1,2} + 2u_{2,2} - u_{3,2} = 0, \\
&u_{1,1} - u_{3,1} + u_{1,2} + u_{3,2} - 2u_{4,2} + 2x_1 = 0, \\
&2u_{0,1} - u_{1,1} - u_{3,1} - u_{1,2} + u_{3,2} + 2x_2 = 0, \\
&u_{1,1} - u_{3,1} + 2u_{0,2} - u_{1,2} - u_{3,2} + 2y_1 = 0, \\
&u_{1,1} + u_{3,1} - 2u_{4,1} - u_{1,2} + u_{3,2} + 2y_2 = 0,
\end{aligned} \tag{4.3}$$

for the system (3.5);

$$\sum_{l=0,1,3,4} 4\partial_{1,i}(u_{l,j}) + \sum_{l=0,1,3,4} (2u_{l,i} - u_{2,i})(2u_{l,j} - u_{2,j}) + 2(x_i y_j + y_i x_j) = 0 \qquad (i, j = 1, 2),$$

for the system (3.6);

$$t_{1,1}\partial_{1,1}(x_i)+t_{1,2}\partial_{1,2}(x_i)+x_i=0, \qquad t_{1,1}\partial_{1,1}(y_i)+t_{1,2}\partial_{1,2}(y_i)+y_i=0,$$
$$t_{1,1}\partial_{1,1}(u_{j,i})+t_{1,2}\partial_{1,2}(u_{j,i})+u_{j,i}=0, \qquad (i=1,2;\ j=0,\ldots,4),$$

for the system (4.2). In the next section, we show that they imply the sixth Painlevé equation.

Under the similarity condition (4.2), the system (3.6) implies

$$2(d_s|U_i)-t_{1,1}(U_i|U_1)-t_{1,2}(U_i|U_2)=0 \qquad (i=1,2).$$

It is expressed in terms of the variables $u_{j,i}$, x_i, y_i as follows:

$$\sum_{l=0,1,3,4} 4u_{l,i}-\sum_{l=0,1,3,4} t_{1,1}(2u_{l,i}-u_{2,i})(2u_{l,1}-u_{2,1})-2t_{1,1}(x_iy_1+y_ix_1)$$
$$-\sum_{l=0,1,3,4} t_{1,2}(2u_{l,i}-u_{2,i})(2u_{l,2}-u_{2,2})-2t_{1,2}(x_iy_2+y_ix_2)=0 \quad (i=1,2). \tag{4.4}$$

Remark 4.1. The systems (3.5) and (4.2) can be regarded as the compatibility condition of the Lax form

$$d_s(\Psi)=M\Psi, \qquad \partial_{1,i}(\Psi)=B_{1,i}\Psi \qquad (i=1,2), \tag{4.5}$$

where $\Psi=W\exp(\xi)$.

5. The sixth Painlevé equation

In the previous section, we have derived the system of the equations

$$\partial_{1,i}(U_j)-\partial_{1,j}(U_i)+[U_j,U_i]=0, \qquad [\Lambda_{1,i},U_j]-[\Lambda_{1,j},U_i]=0,$$
$$(d_s|\partial_{1,i}(U_j))-\tfrac{1}{2}(U_i|U_j)=0, \qquad \sum_{l=1,2} t_{1,l}\partial_{1,l}(U_i)+U_i=0 \qquad (i,j=1,2), \tag{5.1}$$

for the $\mathfrak{g}_0$-valued functions $U_i=U_i(t_{1,1},t_{1,2})$ $(i=1,2)$, as a similarity reduction of the $D_4^{(1)}$ hierarchy of type $s=(1,1,0,1,1)$. In terms of the operators $B_{1,i}=\Lambda_{1,i}+U_i$ and $M=t_{1,1}B_{1,1}+t_{1,2}B_{1,2}$, the system (5.1) is expressed as

$$[\partial_{1,1}-B_{1,1},\partial_{1,2}-B_{1,2}]=0, \qquad [d_s-M,\partial_{1,i}-B_{1,i}]=0 \qquad (i=1,2),$$

with the equations for normalization (3.6). In this section, we show that the sixth Painlevé equation is derived from them.

The operator M is expressed in the form

$$M=\sum_{i=1,2} t_{1,i}\Lambda_{1,i}+\sum_{j=0,1,3,4}\kappa_j\alpha_j^\vee+\eta\alpha_2^\vee+\varphi e_2+\psi f_2,$$

so that

$$\kappa_j=t_{1,1}u_{j,1}+t_{1,2}u_{j,2} \qquad (j=0,1,3,4), \qquad \eta=t_{1,1}u_{2,1}+t_{1,2}u_{2,2},$$
$$\varphi=t_{1,1}x_1+t_{1,2}x_2, \qquad \psi=t_{1,1}y_1+t_{1,2}y_2. \tag{5.2}$$

The system (4.1) implies that the variables κ_j $(j=0,1,3,4)$ are independent of $t_{1,i}$ $(i=1,2)$. Then the following lemma is obtained from (4.3), (4.4) and (5.2).

Lemma 5.1. *The variables $u_{j,i}$, x_i, y_i $(i=1,2;\ j=0,\ldots,4)$ are determined uniquely as polynomials in η, φ and ψ with coefficients in $\mathbb{C}(t_{1,i})[\kappa_j]$. Furthermore, the following relation is satisfied:*

$$\eta^2-(\kappa_0+\kappa_1+\kappa_3+\kappa_4)(\eta+1)+\kappa_0^2+\kappa_1^2+\kappa_3^2+\kappa_4^2+\varphi\psi=0.$$

Thanks to this lemma, the system (5.1) can be rewritten into a system of first-order differential equations for η and φ; we do not give the explicit formulae here.

We denote by $\mathfrak{n}_+$ the subalgebra of $\mathfrak{g}$ generated by e_j $(j = 0, \ldots, 4)$, and by $\mathfrak{b}_+$ the borel subalgebra of $\mathfrak{g}$ defined by $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$. We look for a dependent variable λ such that

$$\widetilde{M} = \exp(-\lambda f_2) M \exp(\lambda f_2) - \exp(-\lambda f_2) d_s(\exp(\lambda f_2)) \in \mathfrak{b}_+,$$
$$\widetilde{B}_{1,i} = \exp(-\lambda f_2) B_{1,i} \exp(\lambda f_2) - \exp(-\lambda f_2) \partial_{1,i}(\exp(\lambda f_2)) \in \mathfrak{b}_+ \qquad (i = 1, 2),$$

namely

$$\varphi\lambda^2 + (2\eta - \kappa_0 - \kappa_1 - \kappa_3 - \kappa_4)\lambda - \psi = 0,$$
$$\partial_{1,i}(\lambda) + x_i\lambda^2 - (u_{0,i} + u_{1,i} - 2u_{2,i} + u_{3,i} + u_{4,i})\lambda - y_i = 0 \qquad (i = 1, 2). \tag{5.3}$$

Note that the definition of $\widetilde{M}$ and $\widetilde{B}_{1,i}$ arises from the gauge transformation $\Psi \to \Phi$ defined by $\Phi = \exp(-\lambda f_2)\Psi$ on the Lax form (4.5). By Lemma 5.1 together with the system (5.1), we can show that

$$\lambda = -\frac{1}{8\varphi}\left(8\eta - \alpha_0^2 - \alpha_1^2 - \alpha_3^2 - \alpha_4^2 + 4\right),$$

satisfies equation (5.3), where α_j $(j = 0, 1, 3, 4)$ are constants defined by

$$\kappa_j = -\tfrac{1}{16}\left(8\alpha_j - \alpha_0^2 - \alpha_1^2 - \alpha_3^2 - \alpha_4^2 - 4\right).$$

We also let μ by a dependent variable defined by $\mu = \varphi$ so that

$$\eta = -\lambda\mu + \tfrac{1}{8}\left(\alpha_0^2 + \alpha_1^2 + \alpha_3^2 + \alpha_4^2 - 4\right), \qquad \varphi = \mu.$$

Then the system (5.1) can be regarded as a system of differential equations for variables λ and μ with parameters α_j $(j = 0, 1, 3, 4)$.

We now regard the system (5.1) as a system of ordinary differential equations with respect to the independent variable $t = t_{1,1}$ by setting $t_{1,2} = 1$. Then the operator $\widetilde{M}$ is written in the form

$$\widetilde{M} = \tfrac{1}{16}\left(\alpha_0^2 + \alpha_1^2 + \alpha_3^2 + \alpha_4^2 - 4\right)K - \sum_{j=0,1,3,4} \frac{1}{2}(\alpha_j - 1)\alpha_j^\vee + F_2 e_2 - F_0 e_0 + (t-1)F_1 e_1$$
$$- (t+1)F_3 e_3 - tF_4 e_4 + e_{20} - (t-1)e_{21} + (t+1)e_{23} + te_{24},$$

where

$$F_0 = \lambda + t, \qquad F_1 = \lambda + \frac{t+1}{t-1}, \qquad F_2 = \mu, \qquad F_3 = \lambda - \frac{t-1}{t+1}, \qquad F_4 = \lambda - \frac{1}{t}.$$

The operator $\widetilde{B} = \widetilde{B}_{1,1}$ is written in the form

$$\widetilde{B} = \widetilde{u}_2 K + \sum_{j=0,1,3,4} \widetilde{u}_j\alpha_j^\vee + \widetilde{x}e_2 - e_0 + (\lambda+1)e_1 - (\lambda-1)e_3 - \lambda e_4 - e_{21} + e_{23} + e_{24},$$

where $\widetilde{u}_2$ is a polynomial in λ, μ and the other coefficients are given by

$$\Theta_0\widetilde{u}_j = F_0F_1F_2F_3F_4F_j^{-1} - \sum_{i=0,1,3,4;i\neq j} \frac{1}{2}(\alpha_i + \alpha_j - 2)F_0F_1F_3F_4F_i^{-1}F_j^{-1}$$
$$- \frac{1}{2}(\alpha_j - 1)F_0(F_0 - F_1 - F_3 - F_4) \qquad (j = 0, 1, 3, 4),$$
$$\Theta_0\widetilde{x} = F_0F_2(F_0 - F_1 - F_3 - F_4) + (\alpha_0 + \alpha_2 - 1)F_0 + (\alpha_1 + \alpha_2 - 1)F_1$$
$$+ (\alpha_3 + \alpha_2 - 1)F_3 + (\alpha_4 + \alpha_2 - 1)F_4,$$

with

$$\Theta_0 = (F_0 - F_1)(F_0 - F_3)(F_0 - F_4), \qquad \alpha_2 = -\tfrac{1}{2}(\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 - 1).$$

Since $\widetilde{M}$ and $\widetilde{B}$ are obtained from M and $B_{1,1}$ by the gauge transformation, they satisfy

$$\left[d_s - \widetilde{M}, \frac{\mathrm{d}}{\mathrm{d}t} - \widetilde{B}\right] = 0.$$

By rewriting this compatibility condition into differential equations for F_j $(j = 0, \ldots, 4)$, we obtain the same system as (1.2), (1.3).

Theorem 5.2. *Under the specialization $t_{1,1} = t$ and $t_{1,2} = 1$, the system (5.1) is equivalent to the sixth Painlevé equation (1.2), (1.3).*

Remark 5.3. The system (1.2), (1.3) can be regarded as the compatibility condition of the Lax pair

$$d_s(\Phi) = \widetilde{M}\Phi, \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}t} = \widetilde{B}\Phi, \tag{5.4}$$

where $\Phi = \exp(-\lambda f_2) W \exp(\xi)$. Let

$$\Omega = \{(1-t)(F_0 - F_1)\}^{-\alpha_1^\vee}\{(1+t)(F_0 - F_3)\}^{-\alpha_3^\vee}\{t(F_0 - F_4)\}^{-\alpha_4^\vee} F_0^{\alpha_2^\vee} \exp(F_0^{-1} e_2)\Phi.$$

Then the system (5.4) is transformed into the Lax pair of the type of [NY3] by the gauge transformation $\Phi \to \Omega$.

Finally, we define the group of symmetries for P_{VI} following [NY2]. Consider the transformations

$$r_i(X) = X\exp(-e_i)\exp(f_i)\exp(-e_i) \qquad (i = 0, \ldots, 4),$$

where

$$X = \exp(\xi)X(0) = W^{-1}Z, \qquad \xi = \sum_{i=1,2}\sum_{k=1,3,\ldots} t_{k,i}\Lambda_{k,i}.$$

Under the similarity condition $\mathcal{M} \in \mathfrak{g}_{\geqslant 0}$, their action on W is given by

$$r_i(W) = \exp(\lambda f_2)\exp\left(\frac{(\alpha_i^\vee | d_s - \widetilde{M})}{(f_i | d_s - \widetilde{M})} f_i\right)\exp(-\lambda f_2)W \qquad (i = 0, 1, 3, 4),$$

$$r_2(W) = W.$$

We also define

$$r_i(\alpha_j) = \alpha_j - \alpha_i a_{ij} \qquad (i, j = 0, \ldots, 4).$$

Then the action of them on the variables λ, μ is described as

$$r_i(F_j) = F_j - \frac{\alpha_i}{F_i} u_{ij} \qquad (i, j = 0, \ldots, 4),$$

where $U = (u_{ij})_{i,j=0}^4$ is the orientation matrix of the Dynkin diagram defined by

$$U = \left\{\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1. \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}\right\}$$

Note that the transformations r_i $(i = 0, \ldots, 4)$ satisfy the fundamental relations for the generators of the affine Weyl group $W\big(D_4^{(1)}\big)$.

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